Analytic Combinatorics Exercise Sheet 3

Exercises for the session on 8/5/2017

Problem 3.1

Let $G(z) = \sum_n G_n z^n$ denote the ordinary generating function for the class of unlabelled plane rooted trees, and let $G(z,u)$ denote the ordinary bivariate generating function for this class, where the second parameter is the degree of the root. Recall that $G(z) = \frac{z}{1-G(z)}$ and $G(z,u) = \frac{z}{1-uG(z)}$.

Show that

$$\frac{\partial}{\partial u} G(z,u)\bigg|_{u=1} = \left(\frac{1}{z} - 1\right) G(z) - 1,$$

and hence that the expected degree of the root when the tree has n vertices is

$$\frac{G_{n+1} - G_n}{G_n}.$$

Recall from Problem 1.1 on Exercise Sheet 1 that

$$G_n = \frac{1}{n}\binom{2n-2}{n-1},$$

and hence show that the expected degree of the root is $\frac{3(n-1)}{n+1}$.

Problem 3.2

Let $T(z)$ denote the exponential generating function for the class of labelled non-plane rooted trees, and let $T(z,u)$ denote the exponential bivariate generating function for this class, where the second parameter is the degree of the root. Recall that $T(z) = ze^{T(z)}$ and $T(z,u) = ze^{uT(z)}$.

Show that the expected degree of the root when the tree has n vertices is $\frac{2(n-1)}{n}$.

Problem 3.3

Let $B(z) = \sum_n B_n z^n$ denote the ordinary generating function for the class of binary strings with no consecutive 0's (note: the empty string is included in this class). Show that

$$B(z) = \frac{1+z}{1-z-z^2},$$

and hence use the result from Problem 1.4 on Exercise Sheet 1 to produce an asymptotic expression for B_n.

A general formula for a meromorphic function $h(z) = \frac{f(z)}{g(z)}$ is given by

$$[z^n]h(z) \sim \frac{(-1)^m m f(\alpha)}{\alpha^m \left.\frac{\mathrm{d}^m g(z)}{\mathrm{d}z^m}\right|_{z=\alpha}} \left(\frac{1}{\alpha}\right)^n n^{m-1}, \tag{1}$$

where α is the pole of $h(z)$ that is closest to the origin (if there is a unique closest pole) and m is the order of α. Evaluate the right-hand-side of (1) when $h(z) = \frac{1+z}{1-z-z^2}$, and check that this agrees with your earlier asymptotic expression for B_n.

Problem 3.4

An *alignment* is a sequence of cycles, and hence has exponential generating function

$$A(z) = \frac{1}{1 - \log \frac{1}{1-z}}.$$

By taking a Taylor expansion of $1 - \log \frac{1}{1-z}$ around the dominant singularity (i.e. the singularity of $A(z)$ that is closest to the origin), show

$$A(z) \sim \frac{-e^{-1}}{z - 1 + e^{-1}},$$

and hence obtain an asymptotic expression for $[z^n]A(z)$.

Evaluate the right-hand-side of (1) when $h(z) = \frac{1}{1-\log \frac{1}{1-z}}$, and check that this agrees with your answer.

Problem 3.5

Let γ_R be the closed contour in the complex plane comprised of the real-line interval $[-R, R]$ and a semi-circle in the upper half-plane of radius R. For $R > 1$, use the residue theorem to calculate

$$\int_{\gamma_R} \frac{1}{1 + x^{2m}} \, \mathrm{d}x$$

for $m \in \mathbb{N}$, and hence show that

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^{2m}} \, \mathrm{d}x = \frac{\pi}{m \sin \frac{\pi}{2m}}.$$

Problem 3.6

Calculate

$$\int_0^{2\pi} \frac{1 - e^{it}}{e^{itn}} \, \mathrm{d}t$$

for $n \in \mathbb{Z}$. Then use Cauchy's Coefficient Formula to show

$$[z^n](1 - z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{it}}{e^{itn}} \, \mathrm{d}t,$$

and use this to check your answer.